研究経過報告書

2023 年　9 月　6 日

研　究　員 （留学者）	所属　理工学部　職　准教授　氏名　新庄　玲子
派 遣 期 間	2022 年　9 月　16 日　～　2023 年　8 月　25 日
研究主題等	図式の補領域に着目した結び目の研究
報 告 事 項	（研究活動の概要、内容、成果等、添付書類の見出し等） 申請時に始めに考えていた受け入れ研究者は，Williams CollegeのColin Adams氏であり，派遣期間中は論文[1]の拡張および，それに関連した結び目図式の補領域に関する研究に取り組みたいと考えていた。 しかし，コロナウイルス感染拡大の状況が不透明だったため,海外への派遣を断念し国内への派遣を選択した。在外研修期間中は、国内における結び目理論の第一人者の一人である早稲田大学教育学部の谷山公規氏に受け入れていただき「図式の補領域に着目した結び目の研究」に従事した。 派遣期間中は，論文[1]で確立した図式の補領域に着目した研究を進めることを目標に，文献調査や研究打ち合わせなどの研究活動を行った。 数年の間，コロナウイルス感染拡大の影響で 対面での研究集会やセミナーへの参加，研究打ち合わせができておらず研究も滞っていたが，対面での研究打ち合わせやセミナーを行えたことは研究の進展に大きな影響を与えた。 研究成果としては論文[2]を執筆することができた。また，派遣期間終了後の27日ではあるが，プレプリントサーバーに投稿することができた。以下，研究成果を具体的に述べる。 結び目図式は，図式が描かれている2 次元球面をいくつかの多辺形領域に

分割する。このとき,各多辺形領域を図式の補領域と呼ぶ。

結び目図式の補領域に着目した結び目の研究として, 結び目群のDehn 表示は古くから知られていたが, その他には活発に研究されていない状況であった。申請者らによる論文[1]では, 補領域に現れる多角形の辺数に着目し, 得られた研究の成果をまとめたものである。この論文の発表後, 関連した研究を行う研究者がでてきており, 論文[1]を引用した論文も発表されている。

球面上の結び目射影図の「補領域のｉ辺形の個数たち f_i ($i \neq 4$)は, 球面のオイラー標数から得られる次の等式を満たすことが知られている。

$$\sum_{i=1}^{\infty} (4-i) f_i = 8 \cdots (*)$$

そこで, 等式（＊)を満たす数列を指定した際に,「任意の結び目が, その数列を実現する図式は存在するか」という問題について考察し, $f_1 = f_2 = 0$ の場合に限定してではあるが, 条件を見たす結び目図式の具体的な構成法を与えることで肯定的に解決ができた。また, この結果から, 全ての結び目は3 辺形 8 個といくつかの 4 辺形のみからなる図式を持つことが従う。論文[2]はこれら一連の研究成果をまとめたものである。

この結果は, グラフ理論で古典的に知られている「平面4価グラフの実現問題」の結び目図式版と言えるが, グラフ理論における結果を利用している。また, 与えた構成法はかなり複雑なもので, 構成できる図式の交点数は膨大なものとなる。そのため今後の課題としては ,

(1) $f_1 = f_2 = 0$ でない場合への拡張

(2) 得られる図式の交点数の評価

が残っている。

（1)については, 本結果の証明に利用した論文[3]における平面4価グラフの構成に関するＤａｌ－ＹｏｕｎｇＪｅｏｎｇ氏の定理の拡張が必要となる。

グラフ理論においては $f_1 = f_2 = 0$ という条件は,「ループ」と「多重辺」という特別な辺をどちらも持たないグラフ（単純グラフ）にグラフを限定するための条件である。グラフ理論においては単純グラフのみを扱うケースは多いので自然なものであるが, 結び目理論において $f_1 = f_2 = 0$ という条件は1辺形と2辺形を持たないことに対応し, 非常に不自然な仮定となる。

論文[2]で行った証明はＪｅｏｎｇ氏の構成したグラフを利用しているため, $f_1 = f_2 = 0$ という条件が必要となっている。Jeong氏の結果の拡張も試みているが, まだ解決には至っていない。

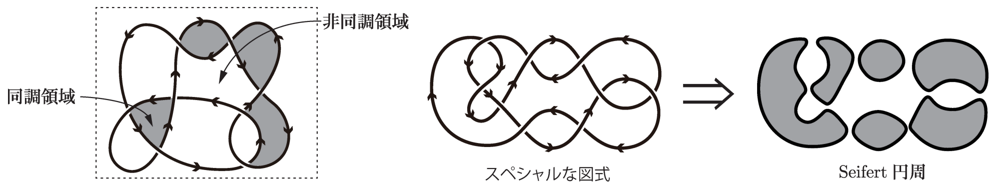
今回得られた結果の $f_1 = f_2 = 0$ という条件を外すために，まずはJeong氏の結果の $f_1 = f_2 = 0$という条件を外すことを目指す。この方向での解決が難しい場合は，グラフ理論の結果を利用しない構成法を与えることに取り組みたい。

(2)に関しては,本構成法で構成した図式は，その結び目図式が表す結び目の最小交点数と比較すると膨大に増えていることは分かるが，どの程度増えているのかはきちんと考察しておらず，「なるべく交点数の少ない図式で実現する」という方向での研究もまだ行っていない。今後は交点数に関する考察も進めていきたい。

また，論文の執筆には至っていないが，結び目図式の補領域の同調領域と非同調領域に関する研究も行った。

有向結び目図式の補領域が同調領域であるとは，その補領域の境界を図式の向きの順に辿れるときにいい,辿れないときは，非同調領域であるという。

結び目の同調数非同調数は，その結び目を表す図式のうちで，スペシャルな図式を全て考え，同調領域，非同調領域の個数の最小値を取ることで定義される。ここで，図式がスペシャルであるというのは，図式のSeifert 円周たちが非交和な円板を張ることを指す(下図参照)。



これら二つの不変量は，派遣者と共同研究者の田中氏により定義されたものである。スペシャルであるという性質は，図式が閉ブレイド状であることの対極にある性質であり,「図式を用いた組合せ論的な結び目理論研究」において近年注目されている。

派遣期間中は，スペシャルな図式の補領域に関する同調領域と非同調領域に着目し，報告者らが定義した同調数と非同調数という2つの不変量の性質をさらに詳しく調べ，結び目理論の具体的な問題へ応用することを目標に研究を進めた。

これらの変量は結び目の複雑さを反映した数量的不変量であり，結び目の標準的種数や組み紐指数などと密接に関係していることが分かってきた。

同調数については，いくつかの結び目に関して不変量の値を決定できた。

また，同調数は常に組み紐指数以上であることが明らかとなっているが,組み紐指数が3以上の結び目については同調数は組み紐指数よりも真に大きい値を取ると予想されるが，その解決には至らなかった。同調数，非同調数についても派遣期間終了後も研究を続けていく予定である。

最後になりますが、報告者を学外派遣研究員として派遣して下さった学校法人国士舘大学および，派遣手続きに尽力してくださった関係者の方々， 派遣中にお世話になった方々に心より感謝いたします。

このような貴重な機会を与えてくださり、ありがとうございました。

今回の在外派遣で得られた経験や知識を活かし，さらなる研究の発展，教育に努めて行きたいと思います。

参考文献

[1] Colin Adams, Reiko Shinjo and Kokoro Tanaka, Complementary regions of knot and link diagrams, Ann. Comb. **15** (2011), no. 4, 549-563

[2] Reiko Shinjo, and Kokoro Tanaka, Any link has a diagram with only triangles and quadrilaterals, preprint, https://arxiv.org/abs/2308.14118

[3] Dal-Young Jeong, Realizations with a cut-through Eulerian circuit, Discrete Math. **137** (1995), no. 1-3, 265-275.

ANY LINK HAS A DIAGRAM WITH ONLY TRIANGLES AND QUADRILATERALS

REIKO SHINJO AND KOKORO TANAKA

Dedicated to Professor Kouki Taniyama on the occasion of his 60th birthday

Abstract. A link diagram can be considered as a 4-valent graph embedded in the 2-sphere and divides the sphere into complementary regions. In this paper, we show that any link has a diagram with only triangles and quadrilaterals. This extends previous results shown by the authors and C. Adams.

1. Introduction

A link diagram can be considered as a 4-valent graph embedded in the 2-sphere and divides the sphere into complementary regions. Given a reduced connected diagram D, let $p_n(D)$ be the number of n-gons of all the complementary regions of D for each $n \geq 1$. Note that $p_1(D) = 0$, since D is reduced. It follows from Euler's formula and some elementary observations that we have the following equation

$$2p_2(D) + p_3(D) = 8 + p_5(D) + 2p_6(D) + 3p_7(D) + \cdots , \tag{1}$$

in which $p_4(D)$ does not appear.

In graph theory, the converse direction has been investigated, dating back to [3]. Grümbaum [4] proved that any sequence $\{p_n\}_{n\geq 2, n\neq 4}$ of nonnegative integers with $p_2 = 0$ that satisfies Equation (1) can be realized as a planar 4-valent 3-connected graph such that the number of its n-gon regions is p_n for all $n \neq 4$. This theorem is known as Eberhard's theorem. We note that the condition $p_2 = 0$ is a typical assumption in graph theory, since the 1-skeletons of convex polytopes are of interest as these graphs on the 2-sphere. We also note that a convex 3-polytopal 4-valent graph is nothing but a connected reduced link projection without bigons. Then Jeong [5] extended this result to show that the resulting graph can be taken as a knot projection rather than a link projection.

In this paper, we investigate such a problem for link diagrams; which sequence $\{p_n\}_{n\geq 2, n\neq 4}$ of nonnegative integers that satisfies Equation (1) can be realized as a diagram of every link such that the number of its n-gon regions is p_n for all $n \neq 4$? This is a continuation of the study of complementary regions of knot and link diagrams by the authors and Adams in [2]. In that paper, we introduced and investigated the notion of universal sequences for knots and links, which will be reviewed and discussed in Section 3 of this paper. See the original paper [2] or the survey [1, Chapter 10] for more details about universal sequences. Main purpose of this paper is to show the following theorem and its corollary.

Date: August 19, 2023.

2020 *Mathematics Subject Classification.* Primary 57K10.

Key words and phrases. Knot diagram; complementary region; 4-valent graph.

Theorem 1.1. *Any sequence* $\{p_n\}_{n\geq 2, n\neq 4}$ *of nonnegative integers with* $p_2 = 0$ *that satisfies Equation* (1) *and any link* L*, there exists a diagram* D_L *of* L *such that* $p_n(D_L) = p_n$ *for all* $n \neq 4$*.*

Since the sequence $p_2 = 0$, $p_3 = 8$ and $p_n = 0$ $(n \geq 5)$ satisfies the assumption of Theorem 1.1, we have the following, which is the title of this paper:

Corollary 1.2. *Any link has a diagram with only eight triangles and quadrilaterals.*

Theorem 1.1 follows from Theorem 1.3, whose proof will be given in Section 2.

Theorem 1.3. *Let* P *be a knot projection with the part as shown in Figure 1, where the numbers from* 0 *to* 2 *indicate the order in which the arcs are traced. Then any link* L *has a diagram* D_L *such that* $p_n(D_L) = p_n(P)$ *for all* $n \neq 4$*.*

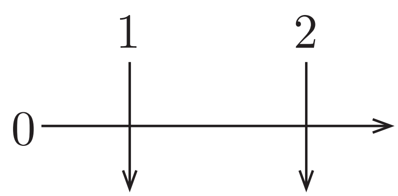


Figure 1. A part of a knot projection consisting of three strands

Proof of Theorem 1.1. Let $\{p_n\}_{n\geq 2, n\neq 4}$ be a sequence of nonnegative integers with $p_2 = 0$ that satisfies Equation (1). It was shown in [5] that there exists a choice of p_4 and a knot projection P such that $p_n(P) = p_n$ for all $n \geq 2$. The proof was inductive and constructive. We recall here a rough outline of the proof in [5]. Start with the knot projection P_0 in Figure 2. It is made up of eight triangles and three quadrilaterals, and has the part as shown in Figure 1; see the right of Figure 2. By performing some local operations for P_0 repeatedly outside the part as shown in Figure 1, the knot projection P_0 can be changed into the desired knot projection P, which also has the the part as shown in Figure 1. Thus Theorem 1.3 can be applied for P and hence any link L has a diagram D_L such that $p_n(D_L) = p_n(P) = p_n$ for all $n \neq 4$. □

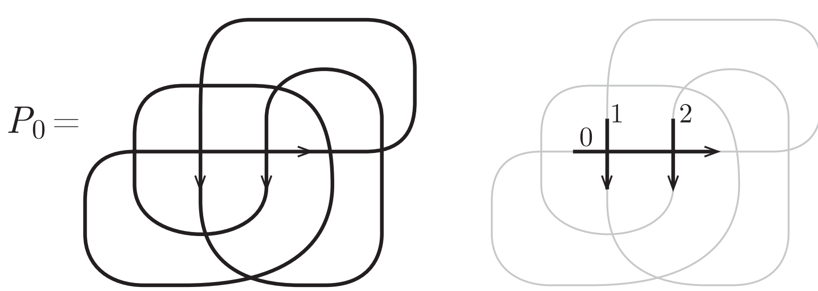


Figure 2. Jeong's knot projection P_0

Although some readers may wonder if it is possible to remove the assumption $p_2 = 0$ from Theorem 1.1, we think that it is an issue in the future. This is because, in order to use Theorem 1.3, we have to extend the result by Jeong [5] without the assumption $p_2 = 0$, however, it may be difficult at this time. Instead, for example, it is possible to prove the following for the sequence $p_2 = 2, p_3 = 4$ and $p_n = 0$ ($n \geq 5$) which satisfies Equation (1).

Proposition 1.4. *Any link has a diagram with only two bigons, four triangles and quadrilaterals.*

Proof. Take a knot projection P_1 as in the left of Figure 6 so that $p_2(P_1) = 2$, $p_3(P_1) = 4$ and $p_n(P_1) = 0$ for all $n \geq 4$. Since P_1 has the part as shown in Figure 1, Theorem 1.3 can be applied for the projection P_1, and hence any link L has a diagram D_L such that $p_n(D_L) = p_n(P_1)$ for all $n \neq 4$, that is, $p_2(D_L) = 2$, $p_3(D_L) = 4$ and $p_n(D_L) = 0$ for all $n \geq 5$. □

Throughout this paper, we use the fact that any diagram can be made a diagram of the unknot by crossing changes. Equivalently, we can make any knot projection into a diagram of the unknot by giving crossing information appropriately. In fact, we can say a little stronger assertion, which will be used later.

Lemma 1.5. *Let Q be a knot projection. If the number of crossings of Q is even (resp. odd), then there exists a diagram D_Q of the unknot whose underlying projection is Q and whose writhe is 0 (resp. $+1$).*

Proof. Induction on the number of crossings. □

2. Proof of Theorem 1.3

Lemma 2.1. *Let P be a knot projection with the part as shown in Figure 1, where the numbers from 0 to 2 in the figure indicate the order in which the arcs are traced. Then there exists a knot projection Q for any integer $N \geq 3$ with the part as shown in Figure 3 such that $p_n(Q) = p_n(P)$ for any $n \neq 4$, where the numbers from 0 to N in the figure also indicate the order in which the arcs are traced.*

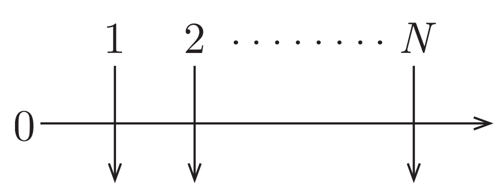


FIGURE 3. A part of a knot projection consisting $(N+1)$-strands

Proof. Let Q_1 be an N-parallel copies of P so that $p_n(Q_1) = p_n(P)$ for all $n \neq 4$. Let Q_2 be a knot projection obtained from Q_1 by replacing a part of Q_1, which corresponds to the N-parallel copies of Figure 3, with the part as in the top of Figure 4, where it depicts the case $N = 3$. When we ignore quadrilaterals, regions of Q_2 are almost identical to those of P. The exact differences are as follows: each of the two shaded regions of Q_2 in the top of Figure 4 has one more edge than the corresponding region in P, and has a newly created triangle next to it. Let Q be a knot projection obtained from Q_2 by creating a pair of two kinks as in the middle of Figure 4, and aligning them along with Q_2 as in the bottom of Figure 4. More precisely, the upper kink goes out from 0 and returns back into 1, and the lower kink goes out from 2 and returns back into 0, where each number from 0 to 2 indicates a set of parallel strands of Q_2 corresponding to the strand labelled by the number in Figure 3. Then we have $p_n(Q) = p_n(P)$ for all $n \neq 4$. Moreover the knot projection Q has the part as in Figure 3; see the bottom right of Figure 4. Hence the knot projection Q is a desired one. □

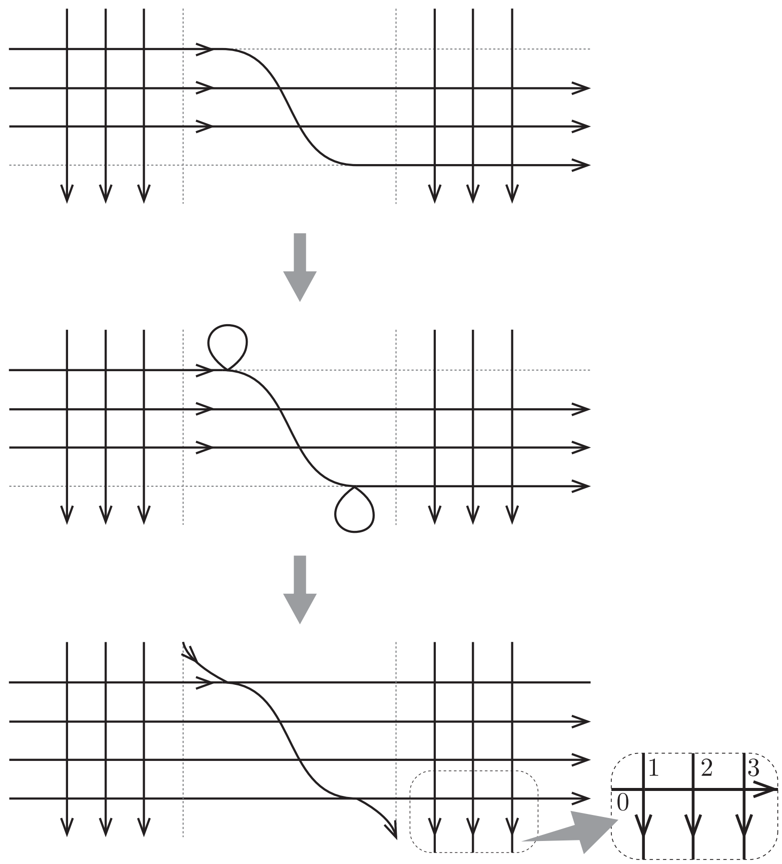


FIGURE 4. How to make Q from P for the case $N = 3$

Proposition 2.2. *Let L be a link, and Q a knot projection with the part as shown in Figure 3 for sufficiently large N depending on L, where the numbers from 0 to N in the figure indicate the order in which the arcs are traced. Then L has a diagram D_L such that $p_n(D_L) = p_n(Q)$ for any $n \neq 4$.*

Proof. Take a closed quasitoric braid diagram D_1 of the link L, where a quasitoric braid is a braid obtained by changing some subset of the crossings in a toric braid. We note that every link can be realized as the closure of a quasitoric braid [6]. When the quasitoric braid diagram D_1 for L is of type (p, q), we take N so that $N > p+q$ and a knot projection Q as in the statement. We discuss the cases where the number of crossings of Q is odd and even.

Case 1. Consider the case where the number of crossings of Q is even. It follows from Lemma 1.5 that there exists a diagram D_Q of the unknot whose underlying projection is Q and whose writhe is 0. Align the closed quasitoric braid diagram D_1 along with the diagram D_Q of the unknot such that quasitoric braid parts are arranged as in the top of Figure 5, where it depicts the case $(p, q) = (3, 2)$, and denote the diagram obtained from D_1 by D_2. Since the writhe of D_Q is zero, the diagram D_2 represents the link L. When we ignore quadrilaterals, regions of D_2 are almost identical to those of Q. The exact differences are as follows: each of $2q$ shaded regions of D_2 in the top of Figure 5 has one more edge than the corresponding region of Q, and and has a newly created triangle next to it. Create q pairs of two kinks for D_2 as in the middle of Figure 5 such that

- the first pair consists of a kink adjacent to the first shaded region from the right on the upper and that adjacent to the first shaded region from the left on the lower,
- the second pair consists of a kink adjacent to the second shaded region from the right on the upper and that adjacent to the second shaded region from the left on the lower,

  ⋮
- the $(q-1)$-st pair consists of a kink adjacent to the $(q-1)$-st shaded region from the right on the upper and that adjacent to the $(q-1)$-st shaded region from the left on the lower,

  and
- the q-th pair consists of a kink adjacent to the q-th shaded region from the right on the upper and that adjacent to the q-th shaded region from the left on the lower.

Then align the totally $2q$ kinks along with D_Q as in the bottom of Figure 5 according to the order such that

- the first kink from the right on the upper leaves from 0, go through from 1 to $q-1$, and returns to q, and then that from the left on the lower leaves from 2, go through from 3 to $q+1$, and returns to 0,
- the second kink from the right on the upper leaves from 0, go through from 1 to $q-2$, and returns to $q-1$, and then that from the left on the lower leaves from 3, go through from 4 to $q+1$, and returns to 0,

  ⋮
- the $(q-1)$-st kink from the right on the upper leaves from 0, go through 1, and returns to 2, and then that from the left on the lower leaves from q, go through $q+1$, and returns to 0,

  and
- the q-th kink from the right on the upper leaves from 0 and returns to 1, and then that from the left on the lower leaves from $q+1$ and returns to 0,

where each number from 0 to $q+1$ indicates a set of parallel strands of D_2 corresponding to the strand labelled by the number in Figure 3. We denote the diagram obtained from D_2 by D_L, where crossing information concerning aligned kinks of D_L must be suitably chosen such that D_L represents the link L, in other words, such that all aligned kinks of D_L can be shrinked back to the original positions as in the middle of Figure 5. Then we have $p_n(D_L) = p_n(Q)$ for all $n \neq 4$, and hence the diagram D_L is a desired one.

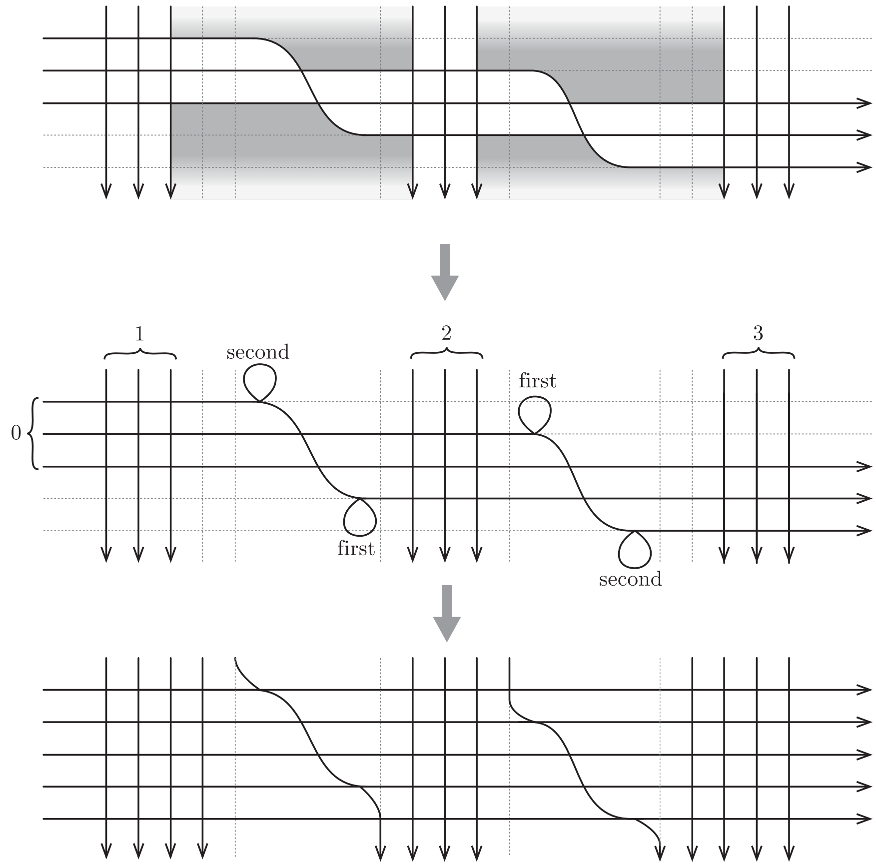


FIGURE 5. How to make D from D_1 for the case $(p, q) = (3, 2)$

Case 2. Consider the case where the number of crossings of Q is odd. It follows from Lemma 1.5 that there exists a diagram D_Q of the unknot whose underlying projection is Q and whose writhe is $+1$. Suppose that the quasitoric braid diagram D of the link L is the closure of a quasitoric braid b_L. Then take a closed quasitoric braid diagram D'_1 as the closure of the product of the two quasitoric braid b_L and $b_{-1} = (\sigma_1\sigma_2 \cdots \sigma_{p-1})^{-p}$. We note that the product of two quasitoric braid is also quaitoric [6]. We remark here that D'_1 does not represent L, since the writhe of D_Q is $+1$. Align D'_1 along with D_Q such that quasitoric braid parts are arranged as in the top of Figure 5, and denote the diagram obtained from D'_1 by D_2. Since the writhe of D_Q is $+1$ and the (quasitoric) braid b_{-1} represents the (-1)-full-twist, the diagram D_2 represents the link L. The rest of the proof is the same as for Case 1 and is therefore omitted. □

Proof of Theorem 1.3. It directly follows from Lemma 2.1 and Proposition 2.2. □

3. Universal sequences

We discuss the relationship with the notion of universal sequences [2] introduce by the authors and Adams. A strictly increasing sequence of integers $(a_1, a_2, a_3, \dots)$

with $a_1 \geq 2$ is said to be *realized* by a link if there exists a diagram for the link such that each complementary region is an a_n-gon for some a_n that appears in the sequence. We note that not every a_n must be realized by a region. We say that a sequence is *universal*[1] if every link has a diagram realizing the sequence. In [2], the following were shown:

- $(n, 2n, 3n, \dots)$ is not universal for any $n \geq 2$ ([2, Theorem 2.3]),
- $(3, 5, 7, \dots)$ is universal ([2, Theorem 3.1]),
- $(3, n, n+1, n+2, \dots)$ is universal for any $n \geq 4$ ([2, Theorem 3.1]),
- $(2, n, n+1, n+2, \dots)$ is universal for any $n \geq 3$ ([2, Theorem 3.1]),
- $(3, 4, n)$ is universal for any $n \geq 5$ ([2, Theorem 3.2 and 3.3]), and
- $(2, 4, 5)$ is universal ([2, Theorem 3.4]).

Using Theorem 1.3, we can extend the last two results and thus give alternative proofs for them. Note that Theorem 3.1 implies the fifth result above.

Theorem 3.1. *The sequence* $(3, 4)$ *is universal.*

Proof. This is a paraphrased assertion of Corollary 1.2. □

Theorem 3.2. *The sequence* $(2, 4, 2k+1)$ *is universal for any* $k \geq 2$.

Proof. Take a knot projection P_k as in the left of Figure 6 for each $k \geq 2$ so that

$$p_2(P_k) = 4k - 2,\ \ p_{2k+1}(P_k) = 4 \ \text{ and } \ p_n(P_k) = 0\ (n \neq 2, 2k+1).$$

Since P_k has the part as shown in Figure 1, Theorem 1.3 can be applied for the projection P_k, and hence any link L has a diagram D_L such that $p_n(D_L) = p_n(P_k)$ for all $n \neq 4$. It follows from $p_n(P_k) = 0$ for all $n \neq 2, 2k+1$ that $p_n(D_L) = 0$ for all $n \neq 2, 4, 2k+1$. This implies that the sequence $(2, 4, 2k+1)$ is universal. □

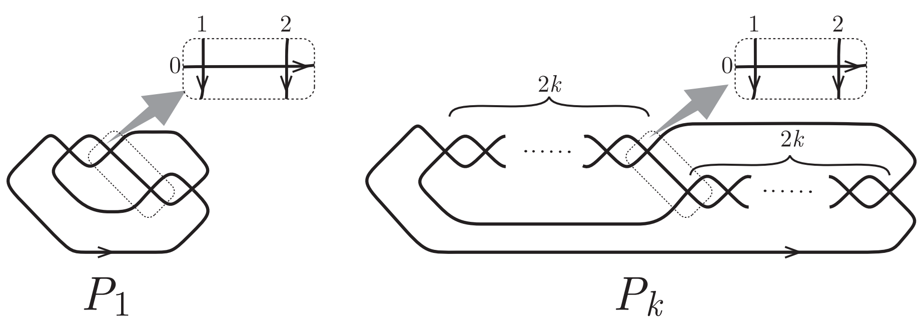


FIGURE 6. The knot projections P_2 and P_k $(k \geq 2)$

Acknowledgments

The second-named author has been supported in part by the Grant-in-Aid for Scientific Research (C), (No. JP17K05242, No. JP21K03220), Japan Society for the Promotion of Science.

---

[1] Although the term "universal for knots and links" is used in [2], the term "universal" is simply used in this paper.

References

1. Colin Adams, Erica Flapan, Allison Henrich, Louis H. Kauffman, Lewis D. Ludwig, and Sam Nelson (eds.), *Encyclopedia of knot theory*, CRC Press, Boca Raton, FL, 2021.
2. Colin Adams, Reiko Shinjo, and Kokoro Tanaka, *Complementary regions of knot and link diagrams*, Ann. Comb. **15** (2011), no. 4, 549–563. MR 2854780
3. Victor Eberhard, *Zur morphologie der polyeder*, Teubner, Leipzig, 1891.
4. B. Grünbaum, *Some analogues of Eberhard's theorem on convex polytopes*, Israel J. Math. **6** (1968), 398–411. MR 244854
5. Dal-Young Jeong, *Realizations with a cut-through Eulerian circuit*, Discrete Math. **137** (1995), no. 1-3, 265–275. MR 1312458
6. V. O. Manturov, *A combinatorial representation of links by quasitoric braids*, European J. Combin. **23** (2002), no. 2, 207–212. MR 1881552

School of Science and Engineering, Kokushikan University, 4-28-1, Setagaya, Setagaya-ku, Tokyo 154-8515, Japan
E-mail address: reiko@kokushikan.ac.jp

Department of Mathematics, Tokyo Gakugei University, Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan
E-mail address: kotanaka@u-gakugei.ac.jp